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RESEARCH ARTICLE

Fixed Point and Common Fixed Point Theorems for Generalized Weak Contraction Mappings of Metric Space

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ABSTRACT

In the present study, we define the common fixed Point theorem for a hybrid generalized multi-valued contraction mapping. Our findings complement, expand upon, harmonize with, and generalize a number of widely-used fixed point theorems published by several authors.

INTRODUCTION

Markin [1] and Nadler [2] started the study of fixed points of multl-valued mappings using the Hausdorff metric. Later, a number of generalizations were obtained, among others, by Ciric [3], Khan [4], Kubiak [5], Reich [6], Smithson [7], and Uegrzyk [8]. The multl-valued contraction principle states that a multi-valued contraction mapping on a complete metric space having values in the set of all closed and bounded subsets of the metric space possesses a fixed point. However, Mukherjee [9], Naimpally et al. [10], Rhoades et al. [1I], and Singh et al. [12] have lately examined hybrid contractions, i.e., contractive conditions involving multi-valued and single-valued mappings. The focus of this work is on a broad category of conditions that involve two multi-valued mappings and a single-valued mapping. We show fixed point and coincidence theorems that enhance, broaden, and consolidate many established fixed point and coincidence theorems. We have compared a few contractive circumstances at the conclusion.

A fascinating field of study is the existence and uniqueness of solutions to functional equations and nonlinear matrix equations. For the existence of the aforementioned equations, metric fixed-point theory offers the most advantageous and effective methods. The writers of [13–18] worked with matrix equations and provided positive definite solutions to show the existence and uniqueness of the answers.

In several fields, including ladder networks [19, 20], control theory [20, 21], and dynamic programming [22–25], matrix equations and functional equations are often generated. Banach [26] has achieved a notable and prosperous outcome, which was subsequently dubbed the Banach contraction principle (BCP). There are several spaces in which the Banach principle has been extended. The Nadler contraction principle (NCP), which applies the Hausdorff metric to set-valued mapping, was further developed and expanded upon by Nadler [27] in 1969.

Subsequently, the BCP was generalized in hundreds of publications published in literature. Very recently, certain writers (see [28–30] and the references therein) demonstrated the contraction principle in metric in controlled metric type spaces where the triangle inequality possesses control functions. Aamri and Moutawakil [31] demonstrate common fixed point solutions under stringent contractive circumstances and defend the (E.A) property for self-mappings, which encompassed the class of compatible and no compatible mappings. In addition to establishing fixed point and coincident point findings with hybrid tight contractions, Kamran [32] illustrated the (E.A) property for hybrid pairs. New common fixed point theorems utilizing hybrid contractive conditions were published by Liu et al. [33] along with the introduction of the common (E.A) property for hybrid pairings of single and multivalued mappings. The concept of the common limit range (CLR) property for single-valued mappings was introduced by Sintunavarat and Kumam [34], who demonstrated its superiority over the property (E.A).

For a hybrid pair of mappings, Imdad et al. [35] developed the common limit range property and showed fixed point results in the symmetric (semimetric) spaces. The ideas were transformed into multivalued mappings by Abbas et al. [25], who also developed linked fixed point and coincidence point theorems that connect hybrid pairs of mappings that meet extended contractive requirements. For hybrid (pair) coupled maps, Deshpande and Handa [36, 37] defined (E.A) property and occasional w-compatibility. They also established common (E.A) property for two hybrid c on traction termed F-contraction.

The F-contraction is a brand-new contraction that Wardowski [38] presented in 2012. In contrast to the established findings of the literature, Wardowski applied the Banach contraction principle (BCP) in a novel method. Sgroi and Vetro [25] investigated multivalued F-contractions in this light and talked about how they may be used for certain integral and functional problems.

Newly, Nashine et al. [39] presented generalized (f, g)-contractions and examined common fixed point outcomes for a hybrid pair under a common limit range property. Their research has implications to Volterra integral inclusion and a specific system of functional equations. Several writers have examined coupled fixed points in various type metric spaces [40–42] for multiple type contraction mappings.

In 1988 and 1989 Kaneko and Sessa were introduced the following definitions:

1. **Weakly commuting:**

Let (ξ, ρ) be a metric space and $\phi: \xi \to LR(\xi)(LR(\xi) = \{V{:}V$ is the set of all non-empty closed and bounded subset of $\xi\ \})$ and $\psi: \xi \to \xi$ are mappings. A set $\{\phi, \psi\}$ is said to be weakly commuting if $\psi\ \phi\ a\ \in LR\ (\xi)$ and $F\ (\phi\ \psi a_l, \psi\ \phi a_l\ ) \to 0$,

whenever $\{a_l\}$ is a sequence in ξ such that $\phi a_l \to A \in LR(\xi)$ and $\psi a_l \to u \in A$ for each $a \in \xi$.

2. **Compatible:**

Let (ξ, ρ) be a metric space and $\phi: \xi \to LR(\xi)$ and $\psi: \xi \to \xi$ are two mappings. A set $\{\phi, \xi\}$ is said to be compatible if and only if

$\psi\phi\ a \in LR(\xi)$ and $F(\phi\ \psi a_l, \psi\ \phi a_l\ ) \to 0$,

whenever $\{a_l\}$ is a sequence in ξ such that $\phi a_l \to A \in LR(\xi)$ and $\psi a_l \to x \in A$ for each $a \in \xi$.

In this paper we extend the great work of author [45],

Theorem 1: Let (ξ, ρ) be a complete metric space and $\phi, \psi: \xi \to LR(\xi)$ and $\psi: \xi \to \xi$ such that

$F(\phi a, \phi_1 a) \leq \frac{\delta[\sigma(\phi a,\psi b)]^2+[\sigma(\phi_1 b,\psi a)]^2}{\sigma(\phi a,\psi b)+\sigma(\phi_1 b,\psi a)} + \theta\delta(\psi a, \psi b)$ holds for all $a, b \in \xi, a \neq b, \phi a \neq \phi b, \phi_1 a, \neq \phi_1 b$;

$\Delta, \theta \geq 0, 2\Delta + \theta < 1$, Whenever $\sigma(\phi a, \psi b) + \sigma(\phi_1 b, \psi a) \neq 0$ and $F(\phi a, \phi_1 a) = 0$ Whenever $\sigma(\phi a, \psi b) + (\phi_1 b, \psi a) = 0$.

Further

(i) $\phi(\xi) \vee \phi_1(\xi) \leq \psi(\xi)$

(ii) $\{\phi, \psi\}$ and $\{\phi_1, \psi\ \}$ are weakly commuting

(iii) ψ is continuous at ξ, then there exists a point r in ξ such that $r = \psi r \in \phi r \cap \phi_1 r$.

Proof: Let us assume that

$$\varepsilon = \frac{\Delta+\vartheta}{1-\Delta}$$

Let $a_0 \in \xi$ and b_1 be arbitrary point ϕa_0.

Also, let $a_1 \in \xi$ such that $b_1 = \psi a_2$ this is possible as $\phi\xi \subseteq \psi\xi$.

Now, using[28], we obtain $b_2 = \in \phi_1 a_1$ such that $\rho(b_1, b_2) \leq F(\phi a_0, \psi a_1) - \varepsilon.\frac{\Delta-1}{\Delta+1}$

We choose $a_2 \in \xi$ such that $b_2 = \psi a_2$ as $\phi_1\xi \subseteq \psi\xi$.

Then we find $b_3 \in \phi a_2$ such that $\rho(b_2, b_3) \leq F(\phi a_2, \psi a_1) + \frac{\Delta-1}{\Delta+1}(-\varepsilon^2)$

In the same way, we select

$$b_{2l} = \psi a_{2l} \quad \in \phi_1 a_{2l-1} \text{ and}$$
$$b_{2l+1} = \psi a_{2l+1} \quad \in \phi a_{2l} \text{ such that}$$

$$\rho(b_{2l+1}, b_{2l+2}) \leq F(\phi a_{2l}, \phi_1 a_{2l-1}) - \frac{\Delta-1}{\Delta+1}(\varepsilon^{2l})$$

Then having selectedb_{2l+1}, take a point $b_{2l+1} = \psi a_{2l+1} \in \phi_1 a_{2l+1}$ such that $\rho(b_{2l+1}, b_{2l+2}) \leq F(\phi_1 a_{2l}, \phi_1 a_{2l+1}) - \frac{\Delta-1}{\Delta+1}(\varepsilon^{2l+1})$

Hence for$l \geq 1$, we have

$$\rho(b_{2l}, b_{2l+1}) \leq F(\phi a_{2l}, \phi_1 a_{2l-1}) - \frac{\Delta-1}{\Delta+1}\varepsilon \text{----------------(1)}$$

$$\leq \varepsilon\frac{\sigma[(\psi a_{2l}, \phi_1 a_{2l-1})]^2 + \sigma[(\psi a_{2l-1}, \phi_1 a_{2l})]^2}{\sigma(\psi a_{2l}, \phi_1 a_{2l-1}) + \sigma(\psi a_{2l-1}, \phi a_{2l})} + \theta\rho(\psi a_{2l-1}, \psi a_{2l}) - \frac{\Delta-1}{\Delta+1}\varepsilon^{2l}$$

$$\leq \varepsilon\sigma[(\psi a_{2l}, \phi_1 a_{2l-1}) + \sigma(\psi a_{2l-1}, \phi a_{2l})] + \theta\rho(\psi a_{2l-1}, \psi a_{2l}) - \frac{\Delta-1}{\Delta+1}\varepsilon^{2l}$$

i.e. $\rho(b_{2l}, b_{2l+1}) \leq \frac{\Delta+\vartheta}{\Delta+1}\ \rho(b_{2l}, b_{2l-1}) + \varepsilon^{2l}$ --------------(2) $=$ $\varepsilon\rho(b_{2l}, b_{2l-1}) + \frac{\varepsilon^{2l}}{\Delta+1}$

Similarly, we can show that

$\rho(b_{2l}, b_{2l-1}) \leq \varepsilon\rho(b_{2l-1}, b_{2l-2}) + \frac{\varepsilon^{2l-1}}{\Delta+1}$ -----------------(3)

Combining equations (2) and (3), we have

$$\rho(b_{l+1}, b_{l+2}) \leq \varepsilon^2\rho(b_l, b_{l-1}) + \frac{2\varepsilon^{l+1}}{\Delta+1} \leq \cdots \leq \varepsilon^l\rho(b_1, b_2) + \frac{l.\varepsilon^{l+1}}{1-\Delta}$$

$\Rightarrow \{b_l\} = \{\psi a_l\}$ Is a Cauchy sequence and by definition it converges to a point r in ξ.

Also, since ψ is a continuous mapping, $\{\psi\psi a_l\}$ converges to a point r inξ.

By definition (1), we have$\psi\phi a_{2l} \in \mathrm{LR}(\xi), a_{2l} \in \xi$, it follows that

$\mathrm{F}(\phi\psi a_{2l}, \psi\phi a_{2l}) \leq \sigma(\phi a_{2l}, \psi a_{2l}) \leq \rho(b_{2l+1}, b_{2l}) \to 0$ as $l \to \infty$

But

$\sigma(\psi \mathrm{r}, \phi\psi a_{2l}) \to 0$ as$l \to \infty$.

So, $\sigma(\psi \mathrm{r}, \phi\psi a_{2l}) \to 0$ as$l \to \infty$.

Again, by using definition,

$\sigma(\psi \mathrm{r}, \phi\psi a_{2l}) \to 0$ as $l \to \infty$

Let$\sigma(\psi \mathrm{r}, \phi \mathrm{r}) \leq \sigma(\psi \mathrm{r}, \phi_1\psi a_{2l-1}) + \mathrm{F}(\phi_1\psi a_{2l-1}, \phi \mathrm{r}) \leq \sigma(\psi \mathrm{r}, \phi_1\psi a_{2l-1}) + \Delta.$

$$\frac{\sigma[(\phi \mathrm{r}, \psi\psi a_{2r-1})]^2 + [\sigma(\phi_1, \psi a_{2r-1}, \psi \mathrm{r})]^2}{\sigma(\phi \mathrm{r}, \psi\psi a_{2r-1}) + \sigma(\phi_1, \psi a_{2r-1}, \psi \mathrm{r})} + \theta\rho(\psi\psi a_{2r-1}, \psi \mathrm{r})$$

Letting$l \to \infty$, then

$\sigma(\psi \mathrm{r}, \phi \mathrm{r}) \leq \varepsilon\sigma(\psi \mathrm{r}, \phi \mathrm{r})$

Hence, $\psi \mathrm{r} \in \phi \mathrm{r}$.

Similarly, one can easily show that $\psi \mathrm{r} \in \phi_1 r$ and therefor;

$$\begin{aligned}
\rho(\psi a_{2r}, \psi \mathrm{r}) &\leq \mathrm{F}(\phi_1\psi a_{2l-1}, \phi \mathrm{r}) \\
&\leq \varepsilon\frac{[\sigma(\psi a_{2l-1}, \phi \mathrm{r})]^2 + [\sigma(\psi \mathrm{r}, \phi_1 a_{2r-1})]^2}{\sigma(\psi a_{2l-1}, \phi \mathrm{r}) + \sigma(\psi \mathrm{r}, \phi_1 a_{2l-1})} + \theta\rho(\psi a_{2l-1}, \psi \mathrm{r}) \\
&\leq \varepsilon\frac{[\sigma(\psi a_{2l-1}, \psi \mathrm{r}) + \sigma(\psi \mathrm{r}, \phi_1 a_{2r-1})]^2}{\sigma(\psi a_{2l-1}, \phi \mathrm{r}) + \sigma(\psi \mathrm{r}, \phi_1 a_{2l-1})} \\
&\leq \varepsilon[\rho(\psi a_{2l-1}, \psi \mathrm{r}) + \rho(\psi \mathrm{r}, \psi a_{2l})] + \theta\rho(\psi a_{2l-1}, \psi \mathrm{r})
\end{aligned}$$

Letting$\to \infty$, then

$\rho(r, \psi \mathrm{r}) \leq (2\Delta + \theta)\rho(r, \psi \mathrm{r})$

Which gives a contradiction. Thus $\mathrm{r} = \psi \mathrm{r}$

$\Rightarrow \mathrm{r} = \psi \mathrm{r} \in \phi \mathrm{r} \cap \phi_1 \mathrm{r}$.

Corollary: -

1. If $\Delta = 0$, we get an extension of famous Banach fixed point theorem.
2. If $\theta = 0$, we get new relation.
3. By the hypothesis, $a \neq$ b, $\phi_1 a \neq \phi_1 b$ is necessary. Since the theorem fails for ϕ and ϕ_1 taken as constant mappings which is expressed by the following example.

   Let $\xi = [0, 1]$,

   $\psi a = 1 - a$.

   We consider $\phi a = \phi_1 a = \{0, 1\}$

   When, $a \in \xi$.

   Thus, by hypothesis,

   $a \neq b,\ \phi a \neq b,\ \psi a \neq \psi b$

   we see that,

   $\psi\left(\frac{1}{2}\right) = 1 - \frac{1}{2} = \frac{1}{2}$

   $\because \frac{1}{2} \nexists \phi\left(\frac{1}{2}\right) \cap \phi_1\left(\frac{1}{2}\right)$.

   $\Rightarrow \phi, \phi_1$, and ψ have no common fixed point.

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